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THE INSTABILITY OF LAGRANGIAN CONFIGURATIONS OF ATTRACTING POINT MASSES†

S. P. SOSNITSKII

Kiev

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The motion of a system of point masses in a negatively uniform force field is investigated. The Lyapunov instability of the Lagrangian configurations of the system, corresponding to constant distances between the points, is established. The proof of the instability is based on representing the Lagrangian of the problem in question in a form which enables the Hamilton action to be calculated in explicit form as a function of the phase variables. Problems of the orbital instability of the Lagrangian configurations are discussed. Copyright © 1996 Elsevier Science Ltd.

1. Consider a system of n attracting point masses, the Lagrangian of which has the form

$$L=T+U=\frac{1}{2}\sum_i m_i\dot{r}_i^2+\lambda\sum_{i<j}\frac{m_im_j}{|r_{ij}|^k},\quad r_{ij}=r_j-r_i \tag{1.1}$$

Here m_i are the masses of the points, $r_i=(x_i, y_i, z_i)^T$ are their radius vectors, referred to an inertial system of coordinates with origin at the centre of the masses m_i, the positive constants λ and k reflect the nature of the attraction, and $i, j = 1, 2, \ldots, n$ everywhere. In particular, if the constant λ is equal to the gravitational constant $G > 0$, and $k = 1$, we arrive at the Newtonian problem of attracting points.

We know [1–3], that the configurations of the point masses $|r_{ij}| = r_{0ij}(t)$ corresponding to periodic motions of the system, discovered by Lagrange in the Newtonian problem, occur in the more general case of Lagrangian (1.1). The situation is similar with the integrals of motion. Below we will essentially use the integral of the energy

$$T-U=h=\text{const} \tag{1.2}$$

and the vector integral of the angular momentum

$$\sum_i m_ir_i\times\dot{r}_i=C \tag{1.3}$$

We will confine ourselves to considering the plane Lagrangian configuration of n point masses

$$|r_{ij}|=r_{0ij}=\text{const} \tag{1.4}$$

assuming, without loss of generality, that the points are situated in the xy plane. Using a system of coordinates rotating round the z axis with constant angular velocity ω, as previously employed in [3, p. 439], the Lagrangian (1.1) can be converted to the form

$$L=T_2+T_1+T_0+U \tag{1.5}$$

$$T_2=\frac{1}{2}\sum_i m_i\dot{r}_i^2,\quad T_1=\omega\sum_i m_i(x_i\dot{y}_i-y_i\dot{x}_i),\quad T_0=\frac{1}{2}\omega^2\sum_i m_i(x_i^2+y_i^2)$$

Here, for convenience, we have retained the previous notation for the components of the vectors r_i and $\dot{r}_i$.

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As a result, the energy integral takes the form

$$T_2 - T_0 - U = h' = \text{const} \tag{1.6}$$

while the projection of the angular momentum onto the z axis can be written in the form

$$\omega^{-1}(T_1 + 2T_0) = C_z' \tag{1.7}$$

It is henceforth more convenient to take integral (1.7) in the form

$$T_1 + 2T_0 = \omega C_z' = c \tag{1.8}$$

The meaning of the use of a rotating system of coordinates is the fact that the Lagrangian configuration (1.4) (periodic motion of the initial system) becomes a set of critical points

$$\dot{r}_i = 0, \ r_i = r_{0i}, \ z_i = 0 \tag{1.9}$$

of the Lagrangian L in the form (1.5). The critical points (1.9) correspond to the position of equilibrium of the system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = 0 \tag{1.10}$$

which simplifies the investigation of the stability of solution (1.4).

Lemma. Lagrangian (1.5) can be represented in the form

$$L = \frac{2}{2-k}\left(\sum_i \frac{\partial L}{\partial \dot{r}_i} r_i\right)^{\bullet} - \frac{2+k}{2-k}(c + h'), \ k \neq 2 \tag{1.11}$$

Proof. Using the equality

$$L = \sum_i p_i \dot{r}_i - H, \ \ p = \frac{\partial L}{\partial \dot{r}_i} \tag{1.12}$$

where H is the Hamiltonian corresponding to Lagrangian (1.5), we can convert (1.12) to the form

$$L = \left(\sum_i p_i r_i\right)^{\bullet} - \sum_i r_i \frac{\partial L}{\partial r_i} - H \tag{1.13}$$

Since

$$-\sum_i r_i \frac{\partial L}{\partial r_i} = -T_1 - 2T_0 + kU = -T_1 - 2T_0 + \frac{k}{2}U + \frac{k}{2}(T_2 - T_0 - h') =$$
$$= -c + \frac{k}{2}(T_2 + T_1 + T_0 + U) - kT_0 - \frac{k}{2}T_1 - \frac{k}{2}h' \tag{1.14}$$

Equation (1.13) can be represented in the form

$$L = \left(\sum_i p_i r_i\right)^{\bullet} - \frac{1}{2}(2+k)(c + h') + \frac{k}{2}L \tag{1.15}$$

whence we obtain (1.11).

If $k = 2$, representation (1.11) will not hold, and we then obtain from (1.15)

$$\left(\sum_i p_i r_i\right)^{\bullet} = \frac{2+k}{2}(c+h') \tag{1.16}$$

Two integrals of motion, which supplement the ten existing ones, follow from (1.16) (compare with [1, p. 294]).

Equation (1.11) is the key equation for investigating the stability of solution (1.4) and, in particular, enables us to obtain an explicit expression for the Hamilton action function for a system with Lagrangian (1.5) (compare with [4])

$$S = \frac{2}{2-k}\sum_i \frac{\partial L}{\partial \dot{r}_i} r_i \Big|_0^t - \frac{2+k}{2-k}(c+h')t, \quad k \neq 2$$

2. Since (1.9) is the position of equilibrium of system (1.5), (1.10), we will represent the quantities r_i in the form

$$r_i = r_{0i} + u_i \tag{2.1}$$

where $u_i = (\xi_i, \eta_i, \zeta_i)^T$ corresponds to a small perturbation of the vector r_{0i}.

From (2.1) we have $\dot{r}_i = \dot{u}_i$. Hence, in the neighbourhood of the equilibrium position the Lagrangian L, by (1.5) and (2.1), takes the form

$$L = (T_0 + U)\big|_{r=r_0} + \dot{\alpha}(\xi,\eta) + T_2^* + T_1^* + T_0^* + U^* \tag{2.2}$$

$$\alpha(\xi,\eta) = \omega\sum_i m_i(x_{0i}\eta_i - y_{0i}\xi_i), \quad T_2^* = \frac{1}{2}\sum_i m_i\dot{u}_i^2$$

$$T_1^* = \omega\sum_i m_i(\xi_i\dot{\eta}_i - \eta_i\dot{\xi}_i), \quad T_0^* = \frac{1}{2}\omega^2\sum_i m_i(\xi_i^2 + \eta_i^2)$$

$$U^* = \frac{1}{2}\sum_{i,j}\frac{\partial^2 U}{\partial r_i \partial r_j}\bigg|_{r=r_0} u_i u_j + O(\|u\|^3), \quad r = (r_1,\ldots,r_n)^T, \quad u = (u_1,\ldots,u_n)^T$$

Using (2.2), we can represent (1.11) in the form

$$L^* = \frac{d}{dt}\Sigma_0 - \frac{2+k}{2-k}(c+h') - (T_0 + U)\big|_{r_i = r_{0i}}, \quad k \neq 2 \tag{2.3}$$

$$\Sigma_0 = \frac{2}{2-k}\sum_i p_i r_i - \alpha(\xi,\eta), \quad L^* = T_2^* + T_1^* + T_0^* + U^* \tag{2.4}$$

Noting that, by (1.14)

$$2T_0\big|_{r_i = r_{0i}} = kU\big|_{r_i = r_{0i}}$$

and, moreover, by (1.6) and (1.8)

$$h' = -(T_0 + U)\big|_{r_i = r_{0i}} - T_0^* - U^* + T_2^*, \quad c = 2T_0\big|_{r_i = r_{0i}} + \Delta(\xi,\eta,\dot{\xi},\dot{\eta})$$

$$\Delta(\xi,\eta,\dot{\xi},\dot{\eta}) = \dot{\alpha}(\xi,\eta) + 2\omega^2\sum_i m_i(x_{0i}\xi_i + y_{0i}\eta_i) + T_1^* + 2T_0^*$$

we can convert (2.3) to the form

$$L^* = \frac{d}{dt}\Sigma_0 - \frac{2+k}{2-k}(c^* + h^*), \quad k \neq 2 \tag{2.5}$$

$$h^* = h' + (T_0 + U)\Big|_{\eta_i = \eta_{0i}} = T_2^* - T_0^* - U^* \tag{2.6}$$

$$c^* = c - 2T_0\Big|_{\eta_i = \eta_{0i}} = \Delta(\xi, \eta, \dot{\xi}, \dot{\eta}) \tag{2.7}$$

Theorem 1. The plane Lagrangian configuration (1.4) of system (1.1) is Lyapunov unstable.

Proof. We first note that if $k = 2$, the instability follows from (1.16). Hence below we will assume that $k \neq 2$.

We will assume that solution (1.4) is Lyapunov stable. Then, the equilibrium position $\dot{u} = u = 0$ ($u = (u_1, \ldots, u_n)^T$, $u_i = (\xi_i, \eta_i, \zeta_i)^T$) of the equations of the perturbed motion

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{u}_i} - \frac{\partial L^*}{\partial u_i} = 0 \tag{2.8}$$

will also be Lyapunov stable.

We will use the analogue of (1.13) for Eqs (2.8)

$$L^* = \left(\sum_i \upsilon_i u_i\right)^{\bullet} - \sum_i u_i \frac{\partial L^*}{\partial u_i} - H^*, \quad \upsilon_i = \frac{\partial L^*}{\partial \dot{u}_i} \tag{2.9}$$

where the Hamiltonian H^* corresponds to the Lagrangian L^*.

Using (2.5) and differentiating (2.9) we obtain

$$\frac{d^2}{dt^2}\Sigma_0 = \left(\sum_i \upsilon_i u_i\right)^{\bullet\bullet} - \left(\sum_i u_i \frac{\partial L^*}{\partial u_i}\right)^{\bullet} \tag{2.10}$$

Integrating (2.10) we obtain

$$\frac{d}{d\tau}\Sigma\Big|_{\tau=t} - \left\{\frac{d}{d\tau}\Sigma + \sum_i u_i \frac{\partial L^*}{\partial u_i}\right\}\Bigg|_{\tau=0} = -\sum_i u_i \frac{\partial L^*}{\partial u_i}\Bigg|_{\tau=t} \tag{2.11}$$

$$\Sigma = \sum_i \left(\frac{2}{2-k} p_i r_i - \upsilon_i u_i\right) - \alpha(\xi, \eta)$$

Since, by (2.1)

$$\sum_i p_i r_i = \sum_i \left[\upsilon_i u_i + m_i(x_{0i}\dot{\xi}_i + y_{0i}\dot{\eta}_i)\right]$$

we can write the expression in braces on the left-hand side of (2.11) in the following form

$$\{\ldots\}\big|_{\tau=0} = -\left\{\dot{\alpha}(\xi, \eta) - \sum_i \left[\frac{k}{2-k}(\upsilon_i u_i)^{\bullet} + \right.\right.$$
$$\left.\left. + \frac{2}{2-k} m_i(x_{0i}\ddot{\xi}_i + y_{0i}\ddot{\eta}_i)\right] - \sum_i u_i \frac{\partial L^*}{\partial u_i}\right\}\Bigg|_{\tau=0} \tag{2.12}$$

Taking into account the fact that, by the equations of perturbed motion (2.8)

$$m_i\ddot{\xi}_i = \frac{\partial(T_0^* + U^*)}{\partial \xi_i} + 2\omega m_i \dot{\eta}_i, \quad m_i \ddot{\eta}_i = \frac{\partial(T_0^* + U^*)}{\partial \eta_i} - 2\omega m_i \dot{\xi}_i \tag{2.13}$$

from (2.9) and (2.11)–(2.13) we obtain

$$\frac{d}{dt}\Sigma+\beta(\xi,\eta,\zeta,\dot{\xi},\dot{\eta},\dot{\zeta})\bigg|_{t=0}=-\sum_i u_i\frac{\partial L^*}{\partial u_i} \tag{2.14}$$

$$\beta=-\frac{2+k}{2-k}\dot{\alpha}-\frac{2}{2-k}\sum_i\left[x_{0i}\frac{\partial(T_0^*+U^*)}{\partial\xi_i}+y_{0i}\frac{\partial(T_0^*+U^*)}{\partial\eta_i}\right]-$$

$$-\frac{k}{2-k}(L^*+H^*)-\frac{2}{2-k}\sum_i u_i\frac{\partial L^*}{\partial u_i} \tag{2.15}$$

The function β is not the integral of motion of system (2.8), which can be seen from the expression

$$\ddot{\alpha}(\xi,\eta)=\omega\sum_i\left[x_{0i}\frac{\partial(T_0^*+U^*)}{\partial\eta_i}-y_{0i}\frac{\partial(T_0^*+U^*)}{\partial\xi_i}-2\omega m_i(x_{0i}\dot{\xi}_i+y_{0i}\dot{\eta}_i)\right]$$

Hence, almost all its trajectories intersect the set of levels of the function β.

According to the assumption on the stability of the equilibrium position $\dot{u}=u=0$, almost all the phase trajectories of system (2.8) which pass through a sufficiently small neighbourhood of it, possess, by virtue of Poincaré's theorem [5], the property of reversibility (Poisson stability). Hence, by considering them to be closed, we can always distinguish, in as small a neighbourhood of the point $\dot{u}=u=0$ as desired, a Poisson stable invariant transitive set Γ with invariant normalized measure μ^* in it [5–7]. Taking into account the fact that β is not the integral of motion of system (2.8), Γ can be chosen in such a way that Γ does not belong to the set of levels of the function β.

By the Birkhoff–Khichin theorem [5, 6] we have

$$\left\langle\sum_i u_i\frac{\partial L^*}{\partial u_i}(\Gamma)\right\rangle=\lim_{t\to\infty}\int_0^t\sum_i u_i\frac{\partial L^*}{\partial u_i}(\Gamma)d\tau=\varkappa=\text{const} \tag{2.16}$$

where the constant $\varkappa$ is independent of the position of the initial point on Γ, with the exception of the set of points of zero measure (with respect to μ^*).

On the other hand, from (2.14) we obtain

$$\left\langle\sum_i u_i\frac{\partial L^*}{\partial u_i}(\Gamma)\right\rangle=-\beta|_{t=0} \tag{2.17}$$

Hence, the mean, according to (2.17), is determined by the initial position of the representative point on Γ. Here, the measure of such initial positions (with respect to μ^*), if we take into account the fact that Γ does not belong to the set of levels of the function β, is non-zero. We have arrived at a contradiction. Hence we conclude that the assumption that the equilibrium $\dot{u}=u=0$ and hence the solution (1.4) is stable is untrue. Theorem 1 is proved.

Corollary. The steady triangular Lagrange solutions $|r_{ij}|=r_0=$ const of the three-body problem are Lyapunov stable.

Notes. 1. The change from (2.10) to (2.14) can be interpreted as the integration procedure in (2.10) of the quantities along the vector field defined by the equations of perturbed motion. Hence, within the framework of the proposed approach, we can follow the relationship with the secondary Lyapunov method, in which, when investigating stability, the differentiation operator along the vector field is employed.

2. The proposed scheme for proving the stability of the Lagrangian configuration (1.4) remains true in the case of the plane problem ($z=0$), when the perturbations belong to the configuration plane.

3. We will confine ourselves below to considering the plane problem of attracting point masses ($z=0$), thereby assuming that the perturbations of the stationary Lagrange configuration (1.4) belong to the configuration plane.

Since system (1.1) allows of the existence of an integral of the mass centre, then in accordance with the choice of the system of coordinates, we can assume without loss of generality that

$$\sum_i m_i r_i = 0 \tag{3.1}$$

and, as a consequence [8, 9], we obtain

$$\sum_i m_i r_i^2 = M^{-1} \sum_{i<j} m_i m_j |r_{ij}|^2, \quad M = \sum_i m_i \tag{3.2}$$

Using (3.1) and (3.2) we have

$$T_0 = \frac{1}{2}\omega^2 M^{-1} \sum_{i<j} m_i m_j |r_{ij}|^2$$

Starting from the existence of solution (1.4) we can write

$$|r_{ij}| = r_{0ij} + x_{ij}, \quad i<j \tag{3.3}$$

where x_{ij} correspond to small perturbations of solution (1.4). Then, in the neighbourhood of the point $|r_{ij}| = r_{0ij}$ we obtain

$$T_0 + U = \sum_{i<j} \left(\frac{1}{2}\omega^2 M^{-1} r_{0ij}^2 + \lambda r_{0ij}^{-k}\right) m_i m_j + \sum_{i<j} (\omega^2 M^{-1} r_{0ij} - \lambda k r_{0ij}^{-k-1}) m_i m_j x_{ij} +$$

$$+\frac{1}{2}\sum_{i<j} \left[\omega^2 M^{-1} + \lambda k(k+1) r_{0ij}^{-k-2}\right] m_i m_j x_{ij}^2 + O\left(\sum_{i<j} \|x_{ij}\|^3\right) \tag{3.4}$$

In particular, as can be seen from (3.4), in the case of the three-body problem, when $r_{0ij} = r_0 = \text{const}$, $\lambda = G, k = 1$, the choice of ω in accordance with the equality $\omega^2 = MGr_0^{-3}$ ensures that there are no linear terms in x_{ij} on the right-hand side of (3.4). Hence, in this case the critical points of the function $T_0 + U$ in the configuration space correspond to its critical points in the mutual-distance space. In the general situation, as has already been seen in the example of the four-body problem [3, p. 428], this is not so.

In addition to (3.3), using (2.1) we have the equation

$$|r_{ij}| = \sqrt{\left[(r_{0j} - r_{0i}) + (u_j - u_i)\right]^2}$$

and hence, the perturbations x_{ij} of the steady mutual distances $|r_{ij}| = r_{0ij}$ can always be expressed as a function of the perturbation u_i and u_j of the vectors r_{0i} and r_{0j}, which correspond to Lagrangian configuration (1.4). The relation between x_{ij} and u_i and u_j enables us to formulate the following question: under what conditions of Lyapunov instability will solution (1.4) imply orbital instability of the latter?

Theorem 2. Suppose the following exist:

1. a departing solution $u^*(t) = (u_1, \ldots, u_n)^T$ of the equations of perturbed motion (2.8), which pass as close to the origin $\dot{u} = u = 0$ as desired;
2. the sequence $\{t_s\} \subset J^+ = [0, a[$ (J^+ is the maximum right interval)

$$\lim_{s\to\infty} t_s = a \quad (s = 0, 1, 2, \ldots)$$

such that the function strictly increases if $s \to \infty$, $u^*(t_s) \in s_\varepsilon = \{u \in R^n, \|u\| < \varepsilon\}$. Then the plane Lagrangian configuration (1.4) is orbitally unstable.

Proof. From (2.2) and (3.4) we have

$$\sum_{i<j} \left(\omega^2 M^{-1} r_{0ij} - \lambda k r_{0ij}^{-k-1}\right) m_i m_j x_{ij} +$$

$$+\frac{1}{2}\sum_{i<j} \left[\omega^2 M^{-1} + \lambda k(k+1) r_{0ij}^{-k-2}\right] m_i m_j x_{ij}^2 + O\left(\sum_{i<j} \|x_{ij}\|^3\right) = T_0^* + U^* \tag{3.5}$$

An increase in $T_0^* + U^*$, by the conditions of Theorem 2 on the departing solution, implies a similar property of the left-hand side of (3.5). Since in the unperturbed motion $|r_{ij}| = r_{0ij}$ = const while $x_{ij}(t_s) = |r_{ij}(t_s)| - r_{0ij}$, an increase in the left-hand side of (3.5) in a certain neighbourhood $s_\delta(\delta = \delta(\varepsilon))$ of the stationary Lagrange configuration (1.4) is equivalent to the perturbed motion emerging from s_δ irrespective of how close it was to the Lagrange configuration (1.4) at the initial instant of time. The property of perturbed motion obtained does not satisfy the definition of orbital stability (see, for example, [10, p. 478]) of the steady motion (1.4). Theorem 2 is proved.

Corollary 1. If the characteristic equation corresponding to the equations in variations of system (2.8) contains roots with real parts, not equal to zero, the plane Lagrange configuration (1.4) is orbitally unstable.

Proof. The equations of perturbed motion (2.8) allow the existence of solutions that are asymptotically attracted to the point $\dot{u} = u = 0$ when $t \to \infty$ and $t \to -\infty$ (see [11, p. 104]). Then, taking the energy integral of the equations of perturbed motion (2.6) into account and noting that the asymptotic solutions belong to the zero-level set of its values, we arrive at the conclusion that the conditions of Theorem 2 are satisfied.

In particular, in the three-body problem the characteristic equation corresponding to the equations in variations, can be reduced to the form [10, p. 586]

$$(\lambda^2 + \omega^2)(\lambda^4 + \omega^2\lambda^2 + k\omega^4) = 0$$

$$k = \frac{27}{4}(m_1m_2 + m_1m_3 + m_2m_3)(m_1 + m_2 + m_3)^{-2}$$

and hence, the stationary triangular Lagrange solutions are orbitally unstable when the following inequality is satisfied

$$27(m_1m_2 + m_1m_3 + m_2m_3) > (m_1 + m_2 + m_3)^2$$

Corollary 2. If a solution of the equations of perturbed motion (2.8) exist, which tends asymptotically to the point $\dot{u} = u = 0$ when $t \to \infty$ $(t \to -\infty)$, the plane Lagrangian configuration (1.4) is orbitally unstable.

Corollaries 1 and 2 enable us to assert that Theorem 2 is constructive.

Notes. 1. The quantity $T_0^* + U^*$ as a function of u is, in many cases, for example, in the three-body problem, degenerate (grad $(T_0^* + U^*)$ may vanish when $\|u\| \neq 0$). Hence, it does not follow from the Lyapunov instability of the equilibrium position $\dot{u} = u = 0$ that a departing solution $u^*(3)$ exists in which the function $T_0^* + U^*$ increases. Hence, without additional limitations, Eq. (3.5) does not enable us to establish the equivalence between the Lyapunov instability of the Lagrange configuration (1.4) and its orbital instability.

2. The equations of perturbed motion (2.8) belong to the class of conservative systems with gyroscopic forces, when the latter are commensurable with the potential forces. Hence, finding the Hamilton action function in explicit form in this case, unlike [4], does not enable one to use it so simply to investigate stability. Nevertheless, the use of the action function as a certain analogue of the auxiliary Lyapunov function also turns out to be constructive in this situation. Note that the well-known criteria of instability of equilibrium (see, for example, review [12]) mainly relate to the case when the potential forces that give rise to the instability of the equilibrium prevail over the gyroscopic forces. When considering the opposite case, when the gyroscopic forces prevail over the potential forces [13], a special structure of the potential energy was assumed, which is not covered by the function $T_0^* + U^*$.

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